

Multiple photon Hamiltonian in linear quantum optical networks

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We give an explicit formula for the Hamiltonian describing the evolution of the quantum state of any number of photons entering a linear optics multiport. The description is based on the Hamiltonian of the optical system for a single photon and comes from relating the evolution in the Lie group that describes the unitary evolution matrices in the Hilbert space of the photon states to the evolution in the Lie algebra of the Hamiltonians for one and multiple photons. We comment some possible applications of our results.

I. LINEAR QUANTUM-OPTICAL NETWORKS

The evolution of the quantum states of light when they pass linear optical networks can be described from the classical scattering matrix of the network S . In classical electromagnetism S relates the amplitudes of the input fields in m input modes with the amplitudes of the m output modes and has many applications in microwave circuit design [1].

In quantum optics, we can replace the field amplitudes with the probability amplitudes in the wavefunction of a photon and use S to see the evolution of the creation operator in each mode.

However, when there are multiple photons, the evolution does not only include wave interference effects, like in classical electromagnetic waves, but also purely quantum effects related to the bosonic nature of the photons. A most striking example is the Hong-Ou-Mandel effect in which two independent photons that reach simultaneously the two separate inputs of a beam splitter always come out together [2]. These interactions have no classical counterpart and are behind the ability of linear optical systems to give an efficient solution to the boson sampling problem, a task that is strongly believed to be inefficient in any classical computing machine [3].

We work with photonic states

$$\bigotimes_{k=1}^m |n_k\rangle_k = |n_1\rangle_1 |n_2\rangle_2 \dots |n_m\rangle_m = |n_1 n_2 \dots n_m\rangle \quad (1)$$

with n photons that are distributed into m orthogonal modes. In the most general case, these modes represent any set of orthogonal single photon states so that

$${}_k \langle 1 | 1 \rangle_l = \delta_{k,l}. \quad (2)$$

The modes can be different paths, which gives a very intuitive picture of the network, but they can also represent orthogonal

temporal wavefunctions, different directions in the same spatial path, photons in orthogonal polarization states, photons with different orbital angular momentum or with a different frequency.

We consider linear optical systems where the number of photons is conserved

$$\sum_{k=1}^m n_k = n. \quad (3)$$

In passive lossless systems the total energy is conserved which fits well with the description in terms of classical fields. The quantum equivalent is a conservation of probability. If we have a superposition of n photon states the output will be a different superposition where all the photon states must sum a probability of one. The input and output states are related by a unitary operator

$$|\psi\rangle_{out} = U |\psi\rangle_{in}. \quad (4)$$

The same mathematical description could be extended to active linear systems as long as the number of photons is preserved. In principle, it could include elements where a frequency change introduces an energy change provided that the total number of photons in the m modes of interest is preserved.

In quantum optics and quantum information we usually find states living in a finite-dimensional Hilbert space and the operators can be written as matrices. Our states live in a Hilbert space \mathcal{H} of a size $M = \binom{m+n-1}{n}$. A generic state $|\psi\rangle$ can be described as a linear combination of the basis elements from Equation (1) that exhaust all the possible ways to distribute n photons into m modes. The problem is equivalent to counting the number of ways to place n balls into m boxes.

A. Unitary evolution

In this finite-dimensional Hilbert space, we can write the states as complex column vectors and the unitary operators U as $M \times M$ unitary matrices. For systems with exactly one

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photon $S = U$ and the quantum state of the photon in mode k , $|1\rangle_k$, is represented as a column vector with m rows that are zero except for the k th row, which carries a one.

For n photons there is a known, more involved, transformation that gives U in terms of S . The evolution matrix U of the system with n photons belongs to the unitary group $U(M)$ and $S \in U(m)$. We can define a group homomorphism $\varphi : U(m) \rightarrow U(M)$.

A nice description of the properties of φ and a verification that it is indeed a homomorphism between groups is given in [3]. We content ourselves with noticing the physically relevant fact that the group operation that gives the composition of operators, matrix multiplication for our description, is preserved. If we have a succession of N optical networks, the first with a matrix S_1 and the last with S_N , the total system has a scattering matrix $S = S_N \dots S_2 S_1$. Their effect of n photons can also be described by multiplying each corresponding unitary operator $U(S) = U(S_N) \dots U(S_2) U(S_1)$ as expected.

Here and in the following sections, we work with the usual photon creation and annihilation operators \hat{a}_k^\dagger and \hat{a}_k [4]. Their effect on states with n_k photons in mode k is

$$\begin{aligned} \hat{a}_k^\dagger |n_k\rangle_k &= \sqrt{n_k + 1} |n_k + 1\rangle_k, \\ \hat{a}_k |n_k\rangle_k &= \sqrt{n_k} |n_k - 1\rangle_k, \quad n \geq 1, \quad \hat{a}_k |0\rangle_k = |0\rangle_k \end{aligned} \quad (5)$$

and have a commutator

$$[\hat{a}_k^\dagger, \hat{a}_j] = \delta_{k,j}. \quad (6)$$

There are a few equivalent ways to write φ . For our purposes, we prefer the description in terms of the evolution of the operators in the Heisenberg picture that shows how all the operators \hat{a}_k^\dagger evolve for all the indices from 1 to m [5]. For any n -photon input state

$$|n_1 n_2 \dots n_m\rangle = \prod_{k=1}^m \left(\frac{\hat{a}_k^{\dagger n_k}}{\sqrt{n_k!}} \right) |00 \dots 0\rangle \quad (7)$$

the output state is given from the elements of S as

$$U |n_1 n_2 \dots n_m\rangle = \prod_{k=1}^m \frac{1}{\sqrt{n_k!}} \left(\sum_{j=1}^m S_{jk} \hat{a}_j^\dagger \right)^{n_k} |00 \dots 0\rangle. \quad (8)$$

Each element of U can be deduced from Equation (8). For an input state $|n_1 n_2 \dots n_m\rangle$ and an output $|n'_1 n'_2 \dots n'_m\rangle$, $\langle n'_1 n'_2 \dots n'_m | U |n_1 n_2 \dots n_m\rangle$ gives the corresponding matrix element for the transition. If we number the states in the basis, for $|q\rangle = |n_1 n_2 \dots n_m\rangle$ and $|p\rangle = |n'_1 n'_2 \dots n'_m\rangle$ (column vectors filled with zeros and a 1 for the q th or p th row respectively), $U_{pq} = \langle p | U | q \rangle$. $|U_{pq}|^2$ is the transition probability from $|q\rangle$ to $|p\rangle$ for the studied optical system. The total probability of finding a photon in an output state $|n_1 n_2 \dots n_m\rangle$ can be interpreted as the Feynman sum of all the possible photon paths that end with the desired number of photons in each mode.

Apart from this description of φ , we can write the elements of U from the permanent of different submatrices of S [6, 7].

B. Effective Hamiltonians

The contribution of this paper is a formula for the evolution to the Hamiltonian. We give the effective Hamiltonian governing the evolution of n photons going through a linear optical system with m modes in terms of the Hamiltonian for the evolution of a single photon. The single photon case has been thoroughly discussed in [8], where it is shown how to write the Hamiltonian for any known optical network made from beam splitters, phase shifters and parametric amplifiers.

The evolution of a quantum state $|\psi(t)\rangle$ with time is the solution of the Schrödinger equation

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = H |\psi(t)\rangle, \quad (9)$$

with a Hamiltonian H and the reduced Planck's constant \hbar . The initial state evolves according to a unitary operator as

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle \quad (10)$$

with $U(t) = e^{-\frac{iHt}{\hbar}}$.

We can absorb $-\hbar$ into the Hamiltonian. Similarly, we can do away with time. Light crosses the whole network and partial evolution is generally not interesting except for some cases in which the fractional depth of a photon into a uniform optical system can play a role equivalent to t . In our finite-dimensional description, the unitary evolution matrix $U = e^{iH_U}$ is the matrix exponential of iH_U for a Hermitian matrix H_U which gives the effective Hamiltonian.

II. THE HAMILTONIAN FOR MULTIPLE PHOTONS

We work with unitary matrices S and U and Hermitian matrices H_S and H_U so that $S = e^{iH_S}$ and $U = e^{iH_U}$. We call S to the unitary $m \times m$ matrix that gives the evolution for a single photon state, which can be identified with the classical scattering matrix of the linear optical system. We call U to the matrix describing the evolution for a general state of n photons.

In terms of group theory, the unitary matrices belong to the unitary group of dimension m and M , with $S \in U(m)$ and $U \in U(M)$, and the Hermitian matrices, up to a factor i , belong to the corresponding associated unitary Lie algebra with $iH_S \in \mathfrak{u}(m)$ and $iH_U \in \mathfrak{u}(M)$. All the results we present can be directly adapted to the alternative description of quantum optics with the special unitary group and its algebra. The unitary matrices would have determinant 1 and the Hermitian matrices would be traceless. The results would hold up to an unmeasurable global phase $e^{i\Phi}$ in S that becomes $e^{in\Phi}$ in U .

We consider $U(m)$ and $U(M)$ as (compact) Lie groups. This means they are compact manifolds and their Lie algebras $\mathfrak{u}(m)$ resp. $\mathfrak{u}(M)$ are the tangent spaces to $U(m)$ resp. $U(M)$ at the identity I_m resp. I_M . Moreover, the exponential map $\mathfrak{u}(M) \rightarrow U(M)$ is well-defined and surjective (cf. [9, Sect. 18.4]).

We want to write the algebra homomorphism that allows us to write H_U in terms of the elements of H_S . First we need to show:

Lemma II.1. *The photonic homomorphism φ is C^∞ .*

Proof. This is trivial, since the entries of $U = \varphi(S)$ are polynomial expressions in the entries of the matrix S . \square

This allows us to prove:

Proposition II.2. *Let $\varphi : U(m) \rightarrow U(M)$ be the photonic homomorphism and consider the differential map $d\varphi : \mathfrak{u}(m) \rightarrow \mathfrak{u}(M)$. The diagram*

$$\begin{array}{ccc} U(m) & \xrightarrow{\varphi} & U(M) \\ \exp \uparrow & & \uparrow \exp \\ \mathfrak{u}(m) & \xrightarrow{d\varphi} & \mathfrak{u}(M) \end{array}$$

is commutative, i.e., $\varphi(\exp(X)) = \exp(d\varphi(X))$ for every $X \in \mathfrak{u}(m)$.

Proof. This follows from Warner [10] together with Lemma II.1. \square

We can now express the differential in terms of creation-annihilation operators as follows:

Proposition II.3. *Let $d\varphi : \mathfrak{u}(m) \rightarrow \mathfrak{u}(M)$ and $H_S = (H_{Sij}) \in \mathfrak{u}(m)$, $|q\rangle = |n_1 n_2 \dots n_m\rangle$ and $|p\rangle = |n'_1 n'_2 \dots n'_m\rangle$ with $p, q \in \mathbb{F}_M$, then*

$$iH_{Upq} = d\varphi(iH_S)_{pq} = \langle p | i \sum_{j=1}^m \sum_{l=1}^m H_{Sjl} \hat{a}_j^\dagger \hat{a}_l | q \rangle \quad (11)$$

for \hat{a}_j^\dagger resp. \hat{a}_k the creation resp. annihilation operator in the j -th resp. l -th mode.

Proof. We work with the differential map and the basis states to find

$$iH_U |n_1 n_2 \dots n_m\rangle = d\varphi(iH_S) |n_1 n_2 \dots n_m\rangle. \quad (12)$$

The elements of H_U are deduced from the effects of $U = \varphi(S) = \varphi(e^{iH_S})$ on the basis states as shown in Eq. (8). We take the elements of S in terms of the exponential of iH_S and work around the identity taking $S = e^{iH_S t}$ so that $S = I_m$ and $U = I_M$ for $t = 0$. Then

$$\begin{aligned} iH_U |n_1 n_2 \dots n_m\rangle &= \frac{d}{dt} \varphi(e^{iH_S t}) |n_1 n_2 \dots n_m\rangle \Big|_{t=0} \\ &= \frac{d}{dt} \prod_{k=1}^m \frac{\left(\sum_{j=1}^m e^{iH_S t} \hat{a}_j^\dagger \right)^{n_k}}{\sqrt{n_k!}} |0 \dots 0\rangle \Big|_{t=0} \\ &= \sum_{l=1}^m \frac{n_l}{\sqrt{n_l!}} \frac{d}{dt} \left(\sum_{j=1}^m e^{iH_S t} \hat{a}_j^\dagger \right)^{n_l-1} \Big|_{t=0} \prod_{k \neq l} \frac{\sum_{j=1}^m \delta_{jk} \hat{a}_j^\dagger}{\sqrt{n_k!}} |0 \dots 0\rangle \\ &= \sum_{l=1}^m \left(\sqrt{n_l} \sum_{j=1}^m iH_{Sjl} \hat{a}_j^\dagger \right) \frac{\hat{a}_l^{n_l-1}}{\sqrt{(n_l-1)!}} \prod_{k \neq l} \frac{\hat{a}_k^{n_k}}{\sqrt{n_k!}} |0 \dots 0\rangle \\ &= \sum_{l=1}^m \sqrt{n_l} \sum_{j=1}^m iH_{Sjl} \hat{a}_j^\dagger |n_1 n_2 \dots n_l - 1 \dots n_m\rangle \\ &= \sum_{l=1}^m \sum_{j=1}^m iH_{Sjl} \hat{a}_j^\dagger \hat{a}_l |n_1 n_2 \dots n_l \dots n_m\rangle. \end{aligned} \quad (13)$$

If we number the basis states, when $|q\rangle = |n_1 n_2 \dots n_m\rangle$ and $|p\rangle = |n'_1 n'_2 \dots n'_m\rangle$, the elements of iH_U are

$$\langle p | iH_U | q \rangle = \langle p | \sum_{l=1}^m \sum_{j=1}^m iH_{Sjl} \hat{a}_j^\dagger \hat{a}_l | q \rangle. \quad (14)$$

\square

The evolution can be written as a sum of terms involving a single photon changing its mode, including changes from one mode to itself, where we have the photon number operator $\hat{n}_l = \hat{a}_l^\dagger \hat{a}_l$. We can interpret the evolution in terms of single photon processes from the weighted sum

$$iH_{Upq} = \sum_{l=1}^m \sum_{j=1}^m i \sqrt{(n_j+1)n_l} H_{Sjl} \langle p | n_1 n_2 \dots n_j + 1 \dots n_l - 1 \dots n_m \rangle. \quad (15)$$

There is a contribution only if the input state $|n_1 n_2 \dots n_m\rangle$ is “one photon away” from the output state $|n'_1 n'_2 \dots n'_m\rangle$. The terms of iH_U coming from a different state $\langle p | q \rangle = 0$ can only include one element of H_S . If $\langle p | q \rangle = 0$, we keep all the terms where the photons move to their original mode and we have one photon number operator \hat{n}_l per occupied mode (a sum of H_{Sll} terms multiplied by the corresponding n_l factor).

III. EXAMPLE FOR TWO PHOTONS IN TWO MODES

We can see a simple example of our result for a system with two photons in two modes ($n = m = 2$). For two modes, we define a scattering matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}. \quad (16)$$

and a Hamiltonian

$$H_S = \begin{pmatrix} H_{S11} & H_{S12} \\ H_{S21} & H_{S22} \end{pmatrix}. \quad (17)$$

If we label the available photon states as $|1\rangle = |20\rangle$, $|2\rangle = |02\rangle$ and $|3\rangle = |11\rangle$, using Equation (8) we obtain the unitary evolution matrix

$$U = \begin{pmatrix} S_{11}^2 & S_{12}^2 & \sqrt{2} S_{11} S_{12} \\ S_{21}^2 & S_{22}^2 & \sqrt{2} S_{21} S_{22} \\ \sqrt{2} S_{11} S_{21} & \sqrt{2} S_{12} S_{22} & S_{11} S_{22} + S_{12} S_{21} \end{pmatrix}. \quad (18)$$

From Equation (14), we can give the Hamiltonian H_U in terms of the elements of H_S as

$$H_U = \begin{pmatrix} 2H_{S11} & 0 & \sqrt{2} H_{S12} \\ 0 & 2H_{S22} & \sqrt{2} H_{S21} \\ \sqrt{2} H_{S21} & \sqrt{2} H_{S12} & H_{S11} + H_{S22} \end{pmatrix}. \quad (19)$$

We can check the results for the simple example of the evolution of two photons inside a balanced beam splitter. The scattering matrix, up to a global phase, is

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (20)$$

We can find the corresponding Hamiltonian, either from [8] of computing $H_S = -i\ln(S)$

$$H_S = \begin{pmatrix} 0.46008 & -1.11072 \\ -1.11072 & 2.68152 \end{pmatrix}. \quad (21)$$

The Hamiltonian H_U is

$$H_U = \begin{pmatrix} 0.92016 & 0 & -1.57080 \\ 0 & 5.36304 & -1.57080 \\ -1.57080 & -1.57080 & 3.14160 \end{pmatrix} \quad (22)$$

and we can check the result is correct by computing e^{iH_U} and seeing it is, indeed, the unitary matrix

$$U = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad (23)$$

we expected from Equation (18).

IV. DISCUSSION

We have presented a formula for the Hamiltonian determining the evolution of n photons through an m -mode linear optics multiport that preserves the photon number. The reduced number of available degrees of freedom (the m^2 real parameters from H_S instead of the M^2 parameters of a general $M \times M$

Hermitian matrix) gives a compact way to study the evolution. The matrix has multiple null entries and, as it is a Hermitian matrix, we only need to compute explicitly the upper or lower triangular matrix plus the diagonal. Computing the unitary evolution U still presents some computational challenges. In particular, computing the matrix exponential can be a bottleneck.

Expressing the evolution as an algebra homomorphism can be useful to study linear optical networks. Some quantum optics problems might be easier to tackle with a description in the Lie algebra using the wealth of results from group theory.

Finally, this kind of analysis can also be extended to general linear optics networks where the number of photons is not conserved like in parametric amplifiers [11]. These systems still admit a linear description with a different Lie group of quasi-unitary matrices and the corresponding associated algebra [8]. We are currently investigating into this generalization as a follow-up work.

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